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String junction from non-commutative super Yang-Mills theory

Hiroyuki Hata and Sanefumi Moriyama

Department of Physics, Kyoto University
Kyoto 606-8502, Japan
E-mail: hata@gauge.scphys.kyoto-u.ac.jp,
moriyama@gauge.scphys.kyoto-u.ac.jp

Abstract: We construct a 1/4 BPS soliton solution in $\mathcal{N}=4$ non-commutative super Yang-Mills theory to the first order in the non-commutativity parameter θ_{ij}. We then solve the non-commutative eigenvalue equations for the scalar fields. The Callan-Maldacena interpretation of the eigenvalues precisely reproduces the expected string junction picture: the string junction is tilted against the D3-branes with angle θ_{ij}.

Keywords: Solitons Monopoles and Instantons, D-branes, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Recently non-commutative Yang-Mills theory has attracted much attention because of its origin as an effective theory of strings [1, 2]. In fact, non-commutative Yang-Mills theory arises as a definite limit of the D-brane effective theory obtained from string theory in the presence of a constant Neveu-Schwarz 2-form $B_{\mu\nu}$ background using point-splitting regularization. On the other hand, the D-brane effective theory obtained from string theory in the same situation and with Pauli-Villars regularization is the ordinary Born-Infeld action. Therefore, there must be a relationship between the non-commutative Yang-Mills theory and the ordinary Born-Infeld action [3]. Exploring this relation through the soliton solutions in both theories is an interesting subject.

The Born-Infeld action with a constant B-field background is equivalent to the Born-Infeld action in a uniform magnetic field, and its classical solution representing a D-string attached to a D3-brane was analyzed in [4]. The result shows that the D-string tilts against the D3-brane because of the force balance between the magnetic force and the string tension. In the U(2) non-commutative Yang-Mills case, believing the force balance, we are led to the picture of two parallel D3-branes with a tilted D-string suspended among them [5].

In ref. [6, 7], the monopole solution in non-commutative U(2) Yang-Mills theory was constructed to the first non-trivial order in the non-commutativity parameter θ_{ij}. In order to obtain the string theory picture by the Callan-Maldacena interpretation [8, 9] where we identify a tube-like configuration of a D3-brane as a D-string, we proposed in [6] the non-commutative eigenvalue equation for a matrix-valued fields. From the eigenvalues of the Higgs scalars, we found that the D-string tilts and the result perfectly agrees with the expected one [5]. Some related issues on monopoles in non-commutative Yang-Mills theory are found in [7] for the ADHMN construction, [10] for the T-dual description, and [11] for U(1) Dirac monopoles.

In this paper, we extend the analysis of ref. [6] for the non-commutative U(2) monopole solution to the 1/4 BPS solution in non-commutative U(3) super Yang-Mills theory. Such solutions were constructed for the ordinary super Yang-Mills theory in [12, 13, 14, 15], and they gave the string junction interpretations predicted in [16, 17, 18] (see figure 1). Here in this paper, first solving the non-commutative BPS equations and then solving the eigenvalue equations for the scalars, we obtain a configuration of string junction which is tilted against the D3-branes as was expected in [5].

The present paper is also interesting as a testing ground of the non-commutative eigenvalue equation, which was proposed in ref. [6] and was one of the non-trivial points in the analysis there. In fact, the expected tilted D-string picture can never be obtained in the U(2) case if we consider only the ordinary eigenvalues of the scalar field as a 2×2 matrix. Our result here for the 1/4 BPS solution gives another evidence for the validity of our non-commutative eigenvalue equation.

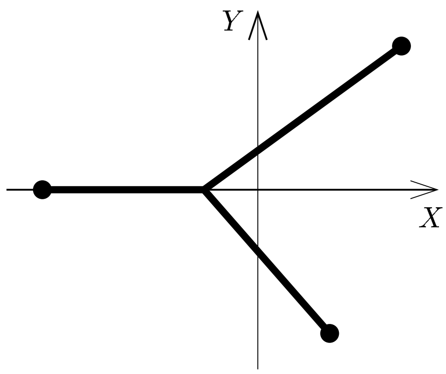

Figure 1: A three-string junction. The blobs at the end of the strings represent D3-branes which are extended in the direction perpendicular to the (X, Y) plane.

The rest of this paper is organized as follows. In section 2, we shall explain the strategy for constructing the string junction solutions in non-commutative Yang-Mills theory. In section 3, as the first step, we solve the non-commutative monopole equation for one of the scalars and the gauge fields to the first order in θ_{ij}, and give a string theory interpretation by solving the non-commutative eigenvalue equation. Next in section 4, we solve the equation for another scalar and obtain its eigenvalues. In section 5, we give the string junction interpretation of the results of sections 3 and 4. In the final section 6, we summarize the paper and give some discussions.

2. Equations for the string junction solution

In this section we shall recapitulate some results in ordinary super Yang-Mills theory necessary for later analysis by generalizing them to the non-commutative case. All the results here can be found in [12, 13, 14, 15] except that we rewrite the ordinary product into the non-commutative star product here.

To construct the string junction solution in the four-dimensional $\mathcal{N} = 4$ noncommutative super Yang-Mills theory with gauge group U(N), we need the bosonic part of the action consisting of the gauge field A_μ and six scalars X^I ($I = 1, \dots, 6$):

$$S = \int d^4x \, \mathrm{Tr} \left(-\frac{1}{4} F_{*\mu\nu} * F_*^{\mu\nu} - \frac{1}{2} D_{*\mu} X^I * D_*^\mu X^I + \frac{1}{4} ([X^I, X^J]_*)_*^2 \right), \tag{2.1}$$

with $F_{*\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_*$ and $D_{*\mu} X \equiv \partial_\mu X - i[A_\mu, X]_*$. The commutator and the square is defined by using the star product: $[A, B]_* \equiv A * B - B * A$ and $(A)_*^2 \equiv A * A$. The star product is defined as usual by

$$(f * g)(x) \equiv f(x) \exp\Big(\frac{i}{2} \theta_{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j} \Big) g(x) = f(x) g(x) + \frac{i}{2} \{f, g\}(x) + O(\theta^2) , \tag{2.2}$$

where $\{f, g\}$ is the Poisson bracket,

$$\{f, g\}(x) \equiv \theta_{ij} \, \partial_i f(x) \, \partial_j g(x) . \tag{2.3}$$

The Gauss law constraint of this system reads

$$D_{*i} E_{*i} = i[X^I, D_{*0} X^I]_* , \tag{2.4}$$

where E_{*i} is the electric field, $E_{*i} = F_{*0i}$.

The energy of this system is given by

$$E = \int d^3x \frac{1}{2} \, \mathrm{Tr} \left((E_{*i})_*^2 + (B_{*i})_*^2 + (D_{*0} X^I)_*^2 + (D_{*i} X^I)_*^2 - \frac{1}{2} ([X^I, X^J]_*)_*^2 \right), \tag{2.5}$$

where B_{*i} is the magnetic field, $B_{*i} = \epsilon_{ijk} F_{*jk}/2$. Hereafter we shall keep only two of the scalar fields, $X^1 = X$ and $X^2 = Y$, nonvanishing. Then, using the Gauss law (2.4), we can rewrite the energy (2.5) into

$$\begin{aligned} E = \int d^3x \frac{1}{2} \, \mathrm{Tr} \Big\{ & (\cos\phi E_{*i} - \sin\phi B_{*i} - D_{*i} X)_*^2 + (\sin\phi E_{*i} + \cos\phi B_{*i} - D_{*i} Y)_*^2 + \\ & + (D_{*0} X + i \sin\phi \, [X, Y]_*)_*^2 + (D_{*0} Y - i \cos\phi \, [X, Y]_*)_*^2 \Big\} + \\ & + (Q_X + M_Y) \cos\phi + (Q_Y - M_X) \sin\phi \\ \geq & \, (Q_X + M_Y) \cos\phi + (Q_Y - M_X) \sin\phi , \end{aligned} \tag{2.6}$$

where ϕ is an arbitrary parameter, and we have $Q_X = \int dS_i \, \mathrm{Tr}(E_{*i} * X)$ and $M_X = \int dS_i \, \mathrm{Tr}(B_{*i} * X)$, and similarly for Q_Y and M_Y. From (2.6) we obtain the classical equations as the condition for saturating the lower bound. We can put $\phi = 0$ without loss of generality since ϕ can be varied by a rotation in the (X, Y) plane. Therefore the equations to be solved are

$$D_{*i} X = E_{*i} , \tag{2.7}$$

$$D_{*i} Y = B_{*i} , \tag{2.8}$$

$$D_{*0} X = 0 , \tag{2.9}$$

$$D_{*0} Y = i \, [X, Y]_* , \tag{2.10}$$

$$D_{*i} D_{*i} X = [Y, [Y, X]_*]_* , \tag{2.11}$$

where the last equation (2.11) is the Gauss law (2.4) with (2.9) and (2.10) substituted.

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Since we are interested in static solutions, we will drop the time-dependence of all the fields. Then, eqs. (2.7), (2.9) and (2.10) are automatically satisfied by putting $A_0 = -X$. The remaining equations we have to solve are eqs. (2.8) and (2.11), which we call non-commutative monopole equation and non-commutative Gauss law, respectively.

In the commutative limit $\theta = 0$, eq. (2.8) reduces to the ordinary BPS monopole equation which was solved in the seminal papers [19, 20] by adopting the spherical symmetry ansatz:

$$A_i^0 = -\frac{\epsilon_{ijk}\widehat{x}_j T_k (K(\xi) - 1)}{r}\,, \qquad Y^0 = -\frac{\widehat{x}_i T_i H(\xi)}{r}\,, \tag{2.12}$$

where the superscripts 0 on A_i and Y denote that they are the 0-th order solution in θ. The dimensionless quantities $\widehat{x}_i$ and ξ are defined by $\widehat{x}_i \equiv x_i/r$ and $\xi \equiv Cr$ using an arbitrary constant C with mass dimension. The matrices T_i $(i = 1, 2, 3)$ are an embedding of SU(2) into the U(N) group: $[T_i, T_j] = i\epsilon_{ijk}T_k$. In the case of the maximal embedding to U(3), the explicit forms of T_i are

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.13}$$

Putting this ansatz into (2.8) with $\theta = 0$, we obtain the equations for K and H,

$$\mathcal{D}K = -HK\,, \qquad \mathcal{D}H = H + 1 - K^2\,, \tag{2.14}$$

where $\mathcal{D}$ denotes the Euler derivative with respect to ξ, $\mathcal{D} \equiv \xi\,(d/d\xi)$. Eq. (2.14) are solved to give

$$K = \frac{\xi}{\sinh\xi}\,, \qquad H = \frac{\xi}{\tanh\xi} - 1\,. \tag{2.15}$$

The behaviors of K and H in the asymptotic region $\xi \to \infty$ are

$$K = O\big(e^{-\xi}\big)\,, \qquad H = \xi - 1 + O\big(e^{-\xi}\big)\,. \tag{2.16}$$

As for the Gauss law (2.11) with $\theta = 0$, the spherical symmetry ansatz,[1]

$$X^0 = \frac{1}{r}\left(\widehat{x}_i T_i P(\xi) + \widehat{x}_i\widehat{x}_j T_{ij}\frac{Q(\xi)}{\xi}\right), \tag{2.17}$$

was considered in [12] where $T_{ij} \equiv \{T_i, T_j\} - \delta_{ij}T_0/3$ with $T_0 \equiv \{T_i, T_i\}$. In the particular case of the maximal embedding into U(N), we have $T_0 = (N^2 - 1)\mathbb{1}/2$.

[1]In the case of the maximal embedding to SU(3), eq. (2.17) is the most general spherically symmetric form for X since T_i and T_{ij} span the whole SU(3). However, for SU(N) with $N \geq 4$, there are other spherically symmetric terms using the symmetric traceless products of T_i's.

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The Gauss law with $\theta = 0$ under the ansatz (2.17) was solved to give

$$\begin{aligned} P &= -\alpha H\,, \\ Q &= -\beta\left(2H^2 + H - 1 + K^2\right), \end{aligned} \tag{2.18}$$

where α and β are arbitrary constants.[2]

In the next two sections we shall solve the non-commutative monopole equation (2.8) and the non-commutative Gauss law (2.11) by the θ expansion. First we shall expand them to the first non-trivial order in θ and solve them by adopting (2.12) and (2.17) as the zero-th order solution.

3. Non-commutative U(3) monopole

In this section, we shall solve the non-commutative monopole equation (2.8) to the first order in θ and evaluate the eigenvalue of the scalar Y for the brane interpretation. By expanding (2.8) to the first order in θ, we get

$$\begin{aligned} \frac{1}{2}\epsilon_{ijk}\Big(\partial_j A_k^1 - \partial_k A_j^1 - i[A_j^0, A_k^1] - i[A_j^1, A_k^0]\Big) - \Big(\partial_i Y^1 - i[A_i^0, Y^1] - i[A_i^1, Y^0]\Big) = \\ = -\frac{1}{2}\epsilon_{ijk}\{A_j^0, A_k^0\} + \frac{1}{2}\{A_i^0, Y^0\} - \frac{1}{2}\{Y^0, A_i^0\}\,, \end{aligned} \tag{3.1}$$

where A_i^1 is the $O(\theta^1)$ part of A_i, namely, $A_i = A_i^0 + A_i^1 + \cdots$, and similarly for Y. Using the zero-th order solution (2.12), we find that the right-hand-side (r.h.s.) of (3.1) is given as a sum of six terms with the following tensor structures concerning θ, the open index i and the $\mathrm{U}(N)$ Lie algebra matrix:

$$\theta_i T_0\,, \qquad \theta_j \widehat{x}_i \widehat{x}_j T_0\,, \qquad \theta_j T_{ij}, \quad \theta_i \widehat{x}_j \widehat{x}_k T_{jk}\,, \qquad \theta_j \widehat{x}_i \widehat{x}_k T_{jk}\,, \qquad \theta_j \widehat{x}_j \widehat{x}_k T_{ik}\,, \tag{3.2}$$

where we have used $\theta_i \equiv \epsilon_{ijk}\theta_{jk}/2$. The coefficient of each quantity in (3.2) is a polynomial of H and K divided by r^4. Apparently, there is another tensor structure $\epsilon_{ijk}\epsilon_{lmn}\theta_l \widehat{x}_j \widehat{x}_m T_{kn}$ which can appear on the r.h.s. of (3.1). However, it is not independent due to the identities,

$$\begin{aligned} \epsilon_{ljk}\widehat{x}_i\widehat{x}_l + \epsilon_{ilk}\widehat{x}_j\widehat{x}_l + \epsilon_{ijl}\widehat{x}_k\widehat{x}_l &= \epsilon_{ijk}\,, \\ \epsilon_{ljk}T_{il} + \epsilon_{ilk}T_{jl} + \epsilon_{ijl}T_{kl} &= 0\,. \end{aligned} \tag{3.3}$$

Using either of them, we can show that $\epsilon_{ijk}\epsilon_{lmn}\theta_l \widehat{x}_j \widehat{x}_m T_{kn} = -\theta_j T_{ij} - \theta_i \widehat{x}_j \widehat{x}_k T_{jk} + \theta_j \widehat{x}_i \widehat{x}_k T_{jk} + \theta_j \widehat{x}_j \widehat{x}_k T_{ik}$.

To solve (3.1), let us adopt the generalized spherical symmetry ansatz [6] for A_i^1 and Y^1. Here the generalized spherical symmetry implies the covariance under the combined rotations of θ_i as well as of x_i and T_i. Noting that all the terms of (3.2) are

[2]Our (α, β) is related to that in ref. [12] by $(\alpha,\beta)_{\mathrm{ref.[12]}} = (-4\alpha, (8/3)\beta)$.

given using either T_0 or T_{ij} for the matrix structure and even numbers of $\widehat{x}_i$, we see that the ansatz for A_i^1 and Y^1 should be given by using T_0 or T_{ij} and odd numbers of $\widehat{x}_i$. For the gauge field A_i^1, at first sight the following seven tensor structures are possible: $\epsilon_{ijk}\theta_j\widehat{x}_kT_0$, $\epsilon_{ijk}\theta_j\widehat{x}_lT_{kl}$, $\epsilon_{ijk}\theta_l\widehat{x}_jT_{kl}$, $\epsilon_{jkl}\theta_j\widehat{x}_kT_{il}$, $\epsilon_{ijk}\theta_j\widehat{x}_k\widehat{x}_l\widehat{x}_mT_{lm}$, $\epsilon_{ijk}\theta_l\widehat{x}_j\widehat{x}_l\widehat{x}_mT_{km}$ and $\epsilon_{jkl}\theta_j\widehat{x}_i\widehat{x}_k\widehat{x}_mT_{lm}$. However, due to the identities (3.3), there are two linear relations among them. Therefore, taking all the independent tensor structures into account, the ansatz is given as follows:

$$\begin{aligned} A_i^1 &= \frac{1}{r^3}\Big(\epsilon_{ijk}\theta_j\widehat{x}_kT_0A(\xi) + \epsilon_{ijk}\theta_j\widehat{x}_lT_{kl}B(\xi) + \epsilon_{jkl}\theta_j\widehat{x}_kT_{il}C(\xi) + \\ &\qquad + \epsilon_{ijk}\theta_j\widehat{x}_k\widehat{x}_l\widehat{x}_mT_{lm}D(\xi) + \epsilon_{ijk}\theta_l\widehat{x}_j\widehat{x}_l\widehat{x}_mT_{km}E(\xi)\Big)\,, \qquad (3.4)\\ Y^1 &= \frac{1}{r^3}\Big(\theta_i\widehat{x}_iT_0\,\mathrm{U}(\xi) + \theta_i\widehat{x}_jT_{ij}V(\xi) + \theta_i\widehat{x}_i\widehat{x}_j\widehat{x}_kT_{jk}W(\xi)\Big)\,. \qquad (3.5)\end{aligned}$$

Putting this ansatz into the l.h.s. of (3.1), we obtain the following system of linear differential equations with inhomogeneous terms:

$$\begin{aligned} \mathcal{D}A-2A-U &= \frac{1}{6}\Big(-H^2K+H(K-1)^2-K(K-1)^2\Big)\,, \qquad (3.6)\\ \mathcal{D}(-A-U)+4A+4U &= \frac{1}{6}\Big(H^2K-H(K-1)(K-3)+(K-1)^3\Big)\,, \qquad (3.7)\\ -\mathcal{D}C+2C+K(-B+C-V)+H(-2C) &= \\ &= \frac{1}{2}(HK+K-1)(H+K-1)\,, \qquad (3.8)\\ \mathcal{D}(B-C+D)-2B+2C-2D-V-W+K(-2B+2C+V)+H(-C) &= \\ &= \frac{1}{2}\Big(H^2K+H(K-1)-(K-1)^2(K+1)\Big)\,, \qquad (3.9)\\ \mathcal{D}(C-V)+B-2C+E+3V+K(-B-2C-2D+V)+H(B+C) &= \\ &= \frac{1}{2}\Big(-H^2K-H(K-1)+(K-1)^2(K+1)\Big)\,, \qquad (3.10)\\ \mathcal{D}(-B+C-E)+3B-3C+2E+V+K(B-C-V-2W)+H(-B+2C-E) &= \\ &= -\frac{1}{2}(HK+2K-2)(H+K-1)\,, \qquad (3.11)\\ \mathcal{D}(-D+E-W)+4D-3E+4W+K(2D-3E+2W)+HE &= \\ &= 0\,. \qquad (3.12)\end{aligned}$$

They are respectively the coefficients of the six structures of (3.2) and of $\theta_j\widehat{x}_i\widehat{x}_j\widehat{x}_k\widehat{x}_lT_{kl}$ (the last one does not appear on the r.h.s. of (3.1)). The first two differential equations (3.6) and (3.7) for A and U are the U(1) parts of the monopole equation (3.1) and decouple from the rest. These are exactly what we solved in the U(2) case [6]:

$$\begin{aligned} A &= \frac{1}{12}(K-1)(2H-K+1)\,,\\ U &= 0\,. \qquad (3.13)\end{aligned}$$

The rest of the equations (3.8)–(3.12) is very complicated and seems hard to solve at first sight. However, we can solve them by assuming that the solutions are given as polynomials of H and K. This polynomial assumption is possible due to the property (2.14) implying that a polynomial of H and K acted by $\mathcal{D}$ is again a polynomial of them. Concretely, we assume that

$$\mathcal{O} = \sum_{n=0}^{N_{\rm max}} \sum_{m=0}^{M_{\rm max}} \mathcal{O}_{nm} H^n K^m \,, \tag{3.14}$$

for the unknown functions $\mathcal{O} = B, C, \ldots$ with suitably large $N_{\rm max}$ and $M_{\rm max}$. Then, using the property (2.14), the differential equations (3.8)–(3.12) are reduced into a set of linear algebraic equations for the coefficients $\mathcal{O}_{nm}$. This set of algebraic equations is easily solved to give

$$\begin{aligned}
B &= -\frac{1}{4} + \frac{1}{4}HK + \frac{1}{4}K^2 + z\mathcal{F} \,, \\
C &= \frac{1}{4} - \frac{3}{4}K - \frac{1}{2}HK + \frac{3}{4}K^2 + \frac{1}{2}HK^2 - \frac{1}{4}K^3 \,, \\
D &= \frac{7}{8} - \frac{1}{8}H - \frac{1}{4}K - \frac{3}{4}HK - \frac{7}{8}K^2 + \frac{1}{4}HK^2 + \frac{1}{4}K^3 + z\mathcal{G} \,, \\
E &= 0 \,, \\
V &= \frac{1}{2} + \frac{1}{4}H - \frac{1}{4}K - \frac{3}{4}HK - \frac{1}{2}K^2 + \frac{1}{4}HK^2 + \frac{1}{4}K^3 - z\mathcal{F} \,, \\
W &= -\frac{7}{8} + \frac{1}{8}H + \frac{1}{4}K + \frac{3}{4}HK + \frac{7}{8}K^2 - \frac{1}{4}HK^2 - \frac{1}{4}K^3 - z\mathcal{G} \,,
\end{aligned} \tag{3.15}$$

with

$$\begin{aligned}
\mathcal{F} &= (-1 + H + 2H^2)K + K^3 \,, \\
\mathcal{G} &= \frac{3}{2}(1 + H) + (1 - H - 2H^2)K - \frac{3}{2}(1 + 2H)K^2 - K^3 \,.
\end{aligned} \tag{3.16}$$

The solution (3.15) does not contain any singularity at the origin $r = 0$ which would invalidate the integration by parts necessary for rewriting the energy (2.5) into (2.6).

In (3.15), the $\mathcal{F}$ and $\mathcal{G}$ terms multiplied by an arbitrary parameter z are a homogeneous solution, namely a solution to the zero-mode equation in the ordinary monopole equation. As we shall see later, this homogeneous solution corresponds to the separation of the monopole centers in the θ_i direction. Besides this zero-mode there is no spherically symmetric homogeneous solution of our interest related to the moduli of the ordinary monopole solution.

Here we should mention the gauge freedom of our solutions. Apparently the set of differential equations (3.6)–(3.12) is underdetermined since the number of unknown functions is one more larger than that of the equations. However, this

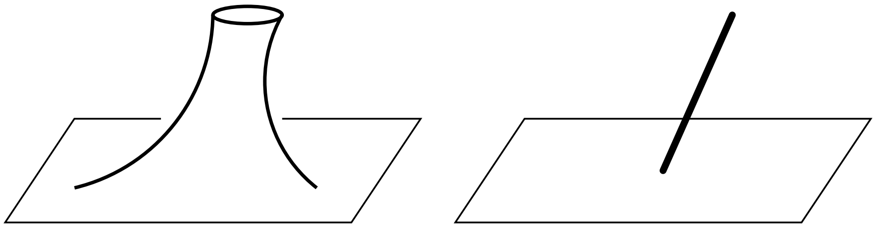

Figure 2: The Callan-Maldacena interpretation. A tube-like deformation of a D3-brane represented by the eigenvalue of the scalar (left) is regarded as a D-string attached to the D3-brane (right).

problem is resolved by noticing that there is a freedom of local gauge transformation, $\delta_\varepsilon A_i = D_{*i}\varepsilon$ and $\delta_\varepsilon X^I = -i[X^I, \varepsilon]_*$, with $\varepsilon = \epsilon_{ijk}\theta_i \widehat{x}_j \widehat{x}_l T_{kl}\, L(\xi)/r^2$, which keeps the generalized spherical symmetry. In the solution (3.15) we have put $E = 0$ by using the freedom of $L(\xi)$.

Having obtained the classical solution to $O(\theta)$, let us next solve the non-commutative eigenvalue equation for the scalar Y in the U(3) case with the maximal SU(2) embedding, and then give the brane interpretation following Callan and Maldacena [8] (see figure 2). The non-commutative eigenvalue equation for a matrix-valued function M proposed in [6] is as follows:

$$M * \boldsymbol{v} = \lambda * \boldsymbol{v}\,, \tag{3.17}$$

where λ is the eigenvalue and $\boldsymbol{v}$ is the eigenvector. Expanding the matrix, the eigenvalue and the eigenvector as

$$M = M^0 + M^1\,, \qquad \lambda = \lambda^0 + \lambda^1\,, \qquad \boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^1\,, \tag{3.18}$$

and plugging them into the eigenvalue equation (3.17), the $O(\theta^1)$ part λ^1 of the eigenvalue is given as [6],

$$\lambda^1 = \frac{i}{2}\boldsymbol{v}^{0\dagger}\{M^0 - \lambda^0 \mathbb{1}, \boldsymbol{v}^0\} + \boldsymbol{v}^{0\dagger} M^1 \boldsymbol{v}^0\,, \tag{3.19}$$

where $\boldsymbol{v}^0$ is normalized, $\boldsymbol{v}^{0\dagger}\boldsymbol{v}^0 = 1$.

The three zero-th order eigenvalues of the scalar Y are

$$\lambda^0_Y = -\frac{H}{r}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \sim \left(-C + \frac{1}{r}\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\,, \tag{3.20}$$

where the last expression is the asymptotic ($r \to \infty$) form obtained by dropping the exponentially decaying terms $O(e^{-\xi})$. Applying (3.19) to the scalar Y with Y^1 given by (3.5), the $O(\theta^1)$ part of the eigenvalues are

$$\lambda_Y^1 = \frac{\theta\cdot\widehat{x}}{r^3}\begin{pmatrix} H/2+4U+(2/3)(V+W) \\ H+4U-(4/3)(V+W) \\ H/2+4U+(2/3)(V+W)\end{pmatrix} \sim \frac{\theta\cdot\widehat{x}}{4r^3}\begin{pmatrix}(3-4z)\xi-4 \\ (2+8z)\xi \\ (3-4z)\xi-4\end{pmatrix}. \qquad (3.21)$$

Here the H term is the contribution of the Poisson bracket in (3.19).

As in the U(2) case of ref. [6], the eigenvalue λ_Y^1 is singular at the origin $r=0$ though the classical solution is regular there. Therefore, we shall restrict our brane analysis to the asymptotic region $r\to\infty$. Summing (3.20) and (3.21), we obtain the total eigenvalues $\lambda_Y = \left(\lambda_Y^{(+)},\lambda_Y^{(0)},\lambda_Y^{(-)}\right)^T$:

$$\begin{aligned}\lambda_Y^{(\pm)} &\sim \mp C \pm \frac{1}{r} + \frac{\theta\cdot\widehat{x}}{4r^3}((3-4z)\xi-4) \\ &= \mp C \pm \left|x_i + \theta_i\left(\pm\left(\frac{1}{4}+z\right)C + \lambda_Y^{(\pm)}\right)\right|^{-1}, \end{aligned}\qquad (3.22)$$

$$\lambda_Y^{(0)} \sim \frac{\theta\cdot\widehat{x}}{2r^3}(1+4z)\xi = \sum_{\pm} \mp\left|x_i \pm \theta_i\left(\frac{1}{4}+z\right)C\right|^{-1}. \qquad (3.23)$$

From (3.22) and (3.23) we can read off the brane configuration representing tilted D-strings suspended among three parallel D3-branes as depicted in figure 3. First, eq. (3.22) implies that, for a given value of $\lambda_Y^{(\pm)}$, the corresponding worldvolume coordinate x_i is located on a sphere with its center at $x_i^{\mathrm{C}(\pm)} = -\theta_i(\pm(1/4+z)C+\lambda_Y^{(\pm)})$. The tilt angle of the D-strings is read off as $-\theta_i$, and the x_i coordinates of the points where the D-strings stick to D3-branes are given as $x_i^{\mathrm{C}(\pm)}$ corresponding to $\lambda_Y^{(\pm)}=\mp C$.

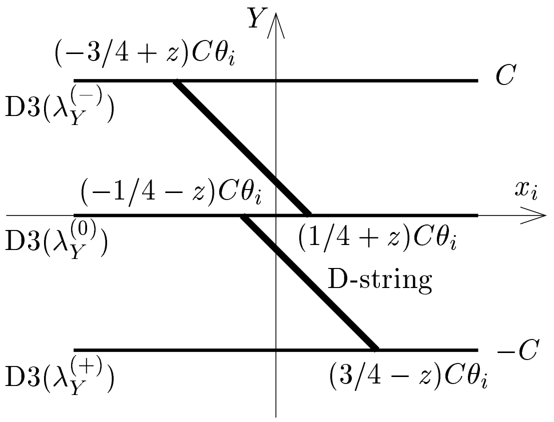

Figure 3: The D-string picture obtained from the asymptotic behavior of the eigenvalues λ_Y. The endpoint x_i coordinates of the D-strings are given on each D3-brane.

Next, from $\lambda_Y^{(0)}$ of (3.23) we see that the middle D3-brane, which was totally flat in the commutative case $\theta=0$ (recall eq. (3.20)), now suffers a deformation with centers at $x_i = \mp(1/4+z)C\theta_i$, which we interpret as the coordinates where the D-strings meet the middle D3-brane. This interpretation is consistent with the D-string picture obtained above from $\lambda_Y^{(\pm)}$. Since, when $\theta=0$, the middle D3-brane was completely flat and did not have any parts identifiable as D-strings, it is impossible to read off the tilt angle of the D-strings from $\lambda_Y^{(0)}$ in the present $O(\theta)$ analysis.

As seen from figure 3, the parameter z corresponds to the relative separation of the two D-strings (namely, the separation of the two monopole centers). Note that the two D-strings are smoothly connected to each other only for a special value of z, $z=-1/4$.

4. Non-commutative Gauss law

Having solved the non-commutative monopole equation (2.8), let us turn to the non-commutative Gauss law (2.11). We shall consider the case of U(3) with the maximal embedding of SU(2). Since the procedure for solving the Gauss law is quite similar to that for the monopole equation (2.8), we shall be brief. Expanding (2.11) to the first order in θ, we have

$$
\begin{aligned}
&\partial_i\left(\partial_i X^1 - i[A_i^0, X^1]\right) - i\left[A_i^0, \partial_i X^1 - i[A_i^0, X^1]\right] - \left[Y^0,[Y^0,X^1]\right] = \\
&\quad = -\partial_i\left(-i[A_i^1, X^0] + \frac{1}{2}\left(\{A_i^0, X^0\} - \{X^0, A_i^0\}\right)\right) + \\
&\qquad + i\left[A_i^1, \partial_i X^0 - i[A_i^0, X^0]\right] + i\left[A_i^0, -i[A_i^1, X^0] + \frac{1}{2}\left(\{A_i^0, X^0\} - \{X^0, A_i^0\}\right)\right] - \\
&\qquad - \frac{1}{2}\left(\left\{A_i^0, \partial_i X^0 - i[A_i^0, X^0]\right\} - \left\{\partial_i X^0 - i[A_i^0, X^0], A_i^0\right\}\right) + \\
&\qquad + i\left[Y^1, -i[Y^0, X^0]\right] + i\left[Y^0, -i[Y^1, X^0] + \frac{1}{2}\left(\{Y^0, X^0\} - \{X^0, Y^0\}\right)\right] - \\
&\qquad - \frac{1}{2}\left(\left\{Y^0, [Y^0, X^0]\right\} - \left\{[Y^0, X^0], Y^0\right\}\right). \qquad (4.1)
\end{aligned}
$$

Due to the monopole equation (2.8) and the Bianchi identity $D_{*i}B_{*i}=0$, $X=\alpha Y$ is a solution to the Gauss law (2.11) for any α. Since the P term in X^0 (2.17) generates this type of solution to the Gauss law, we have only to consider the Q term in (2.17) as X^0 on the r.h.s. of (4.1). We shall put $\beta=1$ in (2.18) for a while for the sake of simplicity. Then, the r.h.s. of (4.1) is evaluated by using the following identities valid for T_i of (2.13):

$$
\begin{aligned}
&\{T_{ij}, T_k\} = \delta_{ik}T_j + \delta_{jk}T_i - \frac{2}{3}\delta_{ij}T_k\,, \\
&[T_{ij}, T_{kl}] = i\left(\delta_{ik}\epsilon_{jlm} + \delta_{il}\epsilon_{jkm} + \delta_{jk}\epsilon_{ilm} + \delta_{jl}\epsilon_{ikm}\right)T_m\,. \qquad (4.2)
\end{aligned}
$$

From the structure of the r.h.s. of (4.1), we see that X^1 consists of terms with one T_i and even numbers of $\widehat{x}_i$. Therefore the ansatz for X^1 is

$$
X^1 = \frac{1}{\xi r^3}\Big(\theta_i T_i R(\xi) + \theta_i \widehat{x}_i \widehat{x}_j T_j S(\xi)\Big)\,. \qquad (4.3)
$$

Note that we have factored out $1/\xi$ in (4.3) similarly to the Q term in (2.17). Putting the solution (2.12) and the ansatz (4.3) into the l.h.s. of (4.1), we obtain the following differential equations for R and S:

$$\begin{aligned}
&\left(\mathcal{D}^2-7\mathcal{D}+11-H^2+2K-K^2\right)R+2KS=\\
&\quad=-\frac{23}{2}-\frac{21}{2}H+\frac{5}{2}H^2+\frac{1}{2}H^3-H^4+\left(20+32H+6H^2-6H^3\right)K+\\
&\qquad+\left(-\frac{19}{2}+9H+17H^2+\frac{23}{2}H^3+3H^4\right)K^2+\left(-4-58H-52H^2-8H^3\right)K^3+\\
&\qquad+\frac{1}{2}\left(35+75H+41H^2\right)K^4+\left(-16-10H\right)K^5+\frac{7}{2}K^6+\\
&\qquad+24z\left\{(-1+3H^2+2H^3)K+2(1+H-2H^2-2H^3)K^3-(1+2H)K^5\right\},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\left(\mathcal{D}^2-7\mathcal{D}+10-2K-2K^2\right)S+\left(-1+H^2+2K-K^2\right)R=\\
&\quad=\frac{1}{2}\left(31+29H-5H^2-H^3+2H^4\right)+\left(-26-34H-2H^2+6H^3\right)K+\\
&\qquad+\left(\frac{19}{2}-23H-23H^2-\frac{15}{2}H^3-3H^4\right)K^2+\left(8+68H+56H^2+8H^3\right)K^3-\\
&\qquad-\frac{1}{2}\left(43+71H+41H^2\right)K^4+\left(18+10H\right)K^5-\frac{7}{2}K^6+\\
&\qquad+24z\Bigg\{\left(1-3H^2-2H^3\right)K+\frac{1}{3}\left(1-2H-3H^2+4H^3+4H^4\right)K^2+\\
&\qquad\qquad+\left(-2-2H+4H^2+4H^3\right)K^3+\frac{2}{3}\left(-1+H+2H^2\right)K^4+\\
&\qquad\qquad+\left(1+2H\right)K^5+\frac{1}{3}K^6\Bigg\}.
\end{aligned} \tag{4.5}$$

Eqs. (4.4) and (4.5) are the coefficient of $\theta_i T_i$ and $\theta_i \widehat{x}_i \widehat{x}_j T_j$ in (4.1), respectively. The solution to these differential equations is again given as polynomials of H and K as in the previous monopole case (3.15):[3]

$$\begin{aligned}
R&=-\frac{3}{2}-\frac{1}{2}H+H^2+\left(2-H-3H^2\right)K+\left(1+\frac{5}{2}H+H^2\right)K^2-\\
&\quad-(2+H)K^3+\frac{1}{2}K^4-8z\left(H^2+H^3\right)K,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
S&=\frac{3}{2}-\frac{1}{2}H-2H^2-\left(2-H-3H^2\right)K-\left(1+\frac{3}{2}H+H^2\right)K^2+(2+H)K^3-\\
&\quad-\frac{1}{2}K^4-8z\left\{\frac{1}{2}(1+H)^2-\left(H^2+H^3\right)K-\left(1+H+H^2\right)K^2+\frac{1}{2}K^4\right\}.
\end{aligned} \tag{4.7}$$

Having obtained the classical solution for X, we shall evaluate its eigenvalues. First, the zero-th order eigenvalues of X are obtained from (2.17) with $(\alpha,\beta)=(0,1)$

[3]There seems to be no physically meaningful homogeneous solution to (4.1) as far as we have examined using the polynomial assumption (3.14).

as

$$\lambda_X^0 = \frac{2Q}{3\xi r}\begin{pmatrix}1\\-2\\1\end{pmatrix} \sim \left(-\frac{4}{3}C + \frac{2}{r}\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}. \tag{4.8}$$

The $O(\theta)$ eigenvalue can be evaluated as in the previous case using (3.19):

$$\lambda_X^1 = \frac{\theta_i \widehat{x}_i}{\xi r^3}\Big(-Q + R + S\Big)\begin{pmatrix}1\\0\\-1\end{pmatrix}. \tag{4.9}$$

Summing (4.8) and (4.9), the total eigenvalues $\lambda_X = \left(\lambda_X^{(+)}, \lambda_X^{(0)}, \lambda_X^{(-)}\right)^T$ of X with $(\alpha, \beta) = (0, 1)$ are given by

$$\begin{aligned}\lambda_X^{(\pm)} &\sim -\frac{4}{3}C + \frac{2}{r} \pm \frac{\theta\cdot\widehat{x}}{r^3}\Big((1-4z)\xi - 2\Big)\\ &= -\frac{4}{3}C + 2\left| x_i \pm \theta_i\left(\frac{1}{2}\lambda_X^{(\pm)} + \left(\frac{1}{6} + 2z\right)C\right)\right|^{-1}, \end{aligned}\tag{4.10}$$

$$\lambda_X^{(0)} \sim \frac{8}{3}C - \frac{4}{r}. \tag{4.11}$$

String junction interpretation of these eigenvalues will be given in the next section.

5. Non-commutative string junction

Now we would like to draw the string junction picture from the asymptotic behavior of the eigenvalues of X and Y via Callan-Maldacena interpretation. First, the string junction picture projected on the (X, Y) plane is the same as in the $\theta = 0$ case (figure 1) since the $O(\theta)$ corrections to the eigenvalues λ_X and λ_Y do not change their leading asymptotic behavior.

Second, the string picture obtained in section 3 from the eigenvalues λ_Y of (3.22) and (3.23) gives the string junction projected on the (Y, x_i) space. Since the string junctions should be connected, we have to take the special value $z = -1/4$ (see figure 4). The three (p,q)-strings constituting the junction look as one straight line tilted against the D3-branes by angle θ_i.

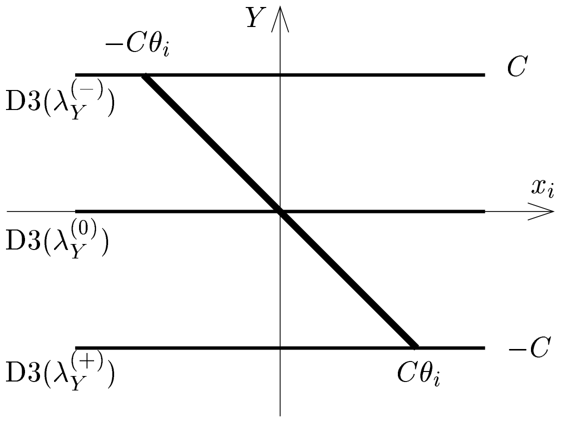

Figure 4: String junction picture in the (Y, x_i) space obtained from the asymptotic behavior of λ_Y. This is the special case of figure 3 with $z = -1/4$. The three (p,q)-strings are seen as one connected straight line.

Before examining the eigenvalues of X for a general (α, β), let us consider the brane interpretation of the eigenvalues (4.10) and (4.11) corresponding to $(\alpha, \beta) = (0, 1)$. In this case, we obtain a three-string junction picture where each of the three strings is attached to the respective D3-brane at the points $x_i = \pm(1 - 4z)C\theta_i/2$ and $x_i = 0$ for the branes corresponding to $\lambda_X^{(\pm)}$ and $\lambda_X^{(0)}$, respectively. Surprisingly, these endpoint x_i coordinates of the three strings in the (X, x_i) space coincide with the corresponding ones in the (Y, x_i) space only when $z = -1/4$, and they are given by $x_i = \pm C\theta_i$ and $x_i = 0$. In the following we shall restrict our arguments to the case $z = -1/4$.

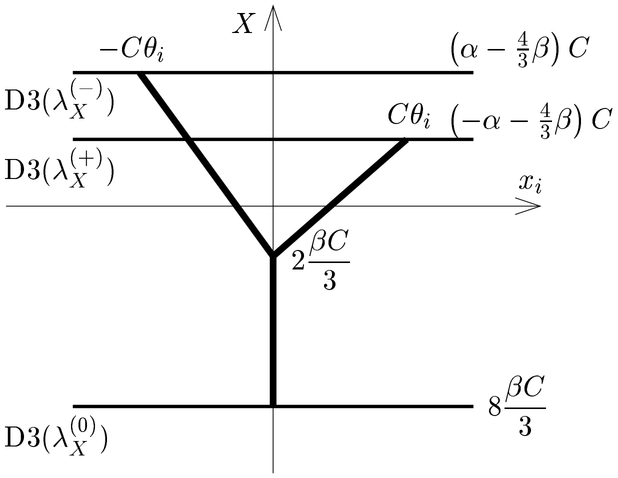

Figure 5: String junction picture in the (X, x_i) space obtained from the asymptotic behavior of λ_X, (5.1) and (5.2), with $z = -1/4$. This figure represents the case with $\alpha > 0$ and $\beta < 0$.

For a general (α, β), the eigenvalues of X are given as $\alpha\lambda_Y + \beta \times [\lambda_X$ of (4.10) and (4.11)]. Explicitly, they are

$$\lambda_X^{(\pm)} \sim \left(\mp\alpha - \frac{4}{3}\beta\right) C + (\pm\alpha + 2\beta) \left| x_i \pm \frac{\theta_i}{\pm\alpha + 2\beta} \left(\lambda_X^{(\pm)} - \frac{2}{3}\beta C\right) \right|^{-1}, \quad (5.1)$$

$$\lambda_X^{(0)} \sim \frac{8}{3}\beta C - \frac{4\beta}{r}\,. \quad (5.2)$$

The string picture of these eigenvalues is given in figure 5.

Let us summarize various quantities of the three-string junction picture obtained here. First, the endpoint coordinates of the three strings in the (X, Y, x_i) space are

$$C\left(\mp\alpha - \frac{4}{3}\beta, \mp 1, \pm\theta_i\right), \qquad C\left(\frac{8}{3}\beta, 0, 0\right), \quad (5.3)$$

for the strings $(\pm)$ and (0), respectively, and the three strings meet at the point

$$C\left(\frac{2\beta}{3}, 0, 0\right). \quad (5.4)$$

Defining the (p, q)-charges of the strings by the leading asymptotic behavior of the eigenvalues of the electric and magnetic fields as

$$(E_i, B_i) \sim \frac{\widehat{x}_i}{2r^2}\,(p, q), \quad (5.5)$$

we have

$$(p, q)^{(\pm)} = (\mp 2\alpha - 4\beta, \mp 2)\,, \qquad (p, q)^{(0)} = (8\beta, 0)\,. \quad (5.6)$$

Then, the tension vectors are given by

$$\vec{T} = (p, q, -q\,\theta_i)\,, \tag{5.7}$$

for each of the three strings, and they are balanced, $\sum \vec{T} = 0$.

As seen from the above analysis, the present string junction with nonvanishing θ is obtained from that with $\theta = 0$ (which is on the $x_i = 0$ plane) by a rotation around the X-axis with angle θ_i (see figure 6). These results are consistent with the expectation obtained from the force balance among the string tension and the magnetic force, which is felt by the charge q at the endpoint of each string in the uniform magnetic field θ_i [5].

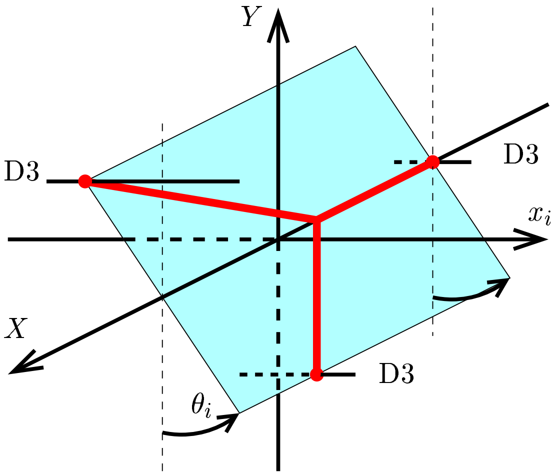

Figure 6: The non-commutative string junction (on the shaded plane) is obtained from the ordinary one with $\theta = 0$ by a rotation around the X-axis with rotation angle θ_i.

6. Summary and discussion

In this paper we have constructed a 1/4 BPS soliton solution in $\mathcal{N} = 4$ non-commutative super Yang-Mills theory. From the asymptotic behavior of the scalar eigenvalues, we have successfully reproduced the expected string junction picture. We would like to emphasize that such consistent eigenvalues can never be obtained without the Poisson bracket term in the eigenvalue formula (3.19). Thus our results give further support of the non-commutative eigenvalue equation (3.17) proposed in ref. [6].

There are a number of questions to be clarified. First, we have chosen the special value $z = -1/4$ as the parameter specifying the separation of the two monopole centers. This led to a consistent string junction picture. However, we have constructed a solution for any value of z, and a question is what the string theory interpretation of our solution for $z \neq -1/4$ is. A similar problem exists already in the 1/4 BPS solution in the ordinary super Yang-Mills theory with the moduli of the separation of the monopole centers [15].

Another question is the simultaneous diagonalizability of the two scalars X and Y. In the case $\theta = 0$, our spherically symmetric solution satisfies $[X^0, Y^0] = 0$ and hence we can consider the eigenvalues of X^0 and Y^0 simultaneously. However, in the non-commutative case, the eigenvectors $\boldsymbol{v}$ of the eigenvalue equation (3.17) are generally different for X and Y. We have to justify the present analysis where we considered the eigenvalues of both X and Y. It is expected that our analysis at the asymptotic region $r \to \infty$ is valid since θ is always multiplied by negative powers of r.

Note added in proofs. In the recent paper hep-th/0002090 by K. Hashimoto and T. Hirayama, our non-commutative string junction solution has been analyzed by transforming it to the commutative description using the Seiberg-Witten map [3]. Their eigenvalues coincide with ours in the asymptotic region, and moreover they are regular at the origin.

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